

# Faithfulness of a functor of Quillen

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## Abstract

There exists a canonical functor from the category of fibrant objects of a model category modulo cylinder homotopy to its homotopy category. We show that this functor is faithful under certain conditions, but not in general.

## §1 Introduction

We let  $\mathcal{M}$  be a model category. QUILLEN defines in [5, ch. I, §1] a homotopy relation on the full subcategory  $\mathbf{Fib}(\mathcal{M})$  of fibrant objects, using cylinders. He obtains a quotient category  $\mathbf{Fib}(\mathcal{M})/\sim$  and a canonical functor

$$\mathbf{Fib}(\mathcal{M})/\sim \rightarrow \mathbf{Ho} \mathbf{Fib}(\mathcal{M}).$$

The question occurs whether this functor is faithful.

We show that it is faithful if  $\mathcal{M}$  is left proper and fulfills an additional technical condition. Moreover, we show by an example that it is not faithful in general.

## Conventions and notations

- The composite of morphisms  $f: X \rightarrow Y$  and  $g: Y \rightarrow Z$  is denoted by  $fg: X \rightarrow Z$ .
- Given  $n \in \mathbb{N}_0$ , we abbreviate  $\mathbb{Z}/n := \mathbb{Z}/n\mathbb{Z}$ . Given  $k, m, n \in \mathbb{N}_0$ , we write  $k: \mathbb{Z}/m \rightarrow \mathbb{Z}/n, a + m\mathbb{Z} \mapsto ka + n\mathbb{Z}$ , provided  $n$  divides  $km$ .
- Given a category  $\mathcal{C}$  with finite coproducts and objects  $X, Y \in \mathbf{Ob} \mathcal{C}$ , we denote by  $X \amalg Y$  a (chosen) coproduct. The embedding  $X \rightarrow X \amalg Y$  is denoted by  $\text{emb}_0$ , the embedding  $Y \rightarrow X \amalg Y$  by  $\text{emb}_1$ . Given morphisms  $f: X \rightarrow Z$  and  $g: Y \rightarrow Z$  in  $\mathcal{C}$ , the induced morphism  $X \amalg Y \rightarrow Z$  is denoted by  $\begin{pmatrix} f \\ g \end{pmatrix}$ .
- Given a category  $\mathcal{C}$  and an object  $X \in \mathbf{Ob} \mathcal{C}$ , the category of objects in  $\mathcal{C}$  under  $X$  will be denoted by  $(X \downarrow \mathcal{C})$ . The objects in  $(X \downarrow \mathcal{C})$  are denoted by  $(Y, f)$ , where  $Y \in \mathbf{Ob} \mathcal{C}$  and  $f: X \rightarrow Y$  is a morphism in  $\mathcal{C}$ .

## §2 Preliminaries from homotopical algebra

We recall some basic facts from homotopical algebra. Our main reference is [5, ch. I, §1].

### Model categories

Throughout this note, we let  $\mathcal{M}$  be a model category, cf. [5, ch. I, §1, def. 1]. In  $\mathcal{M}$ , there are three kinds of distinguished morphisms, called *cofibrations*, *fibrations* and *weak equivalences*. Cofibrations are closed under pushouts. If weak equivalences in  $\mathcal{M}$  are closed under pushouts along cofibrations,  $\mathcal{M}$  is said to be *left proper*, cf. [3, def. 13.1.1(1)].

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An object  $X \in \text{Ob } \mathcal{M}$  is said to be *fibrant* if the unique morphism  $\mathcal{M} \rightarrow *$  is a fibration, where  $*$  is a (chosen) terminal object in  $\mathcal{M}$ . The full subcategory of  $\mathcal{M}$  of fibrant objects is denoted by  $\mathbf{Fib}(\mathcal{M})$ .

The *homotopy category* of  $\mathcal{U} \in \{\mathcal{M}, \mathbf{Fib}(\mathcal{M})\}$  is a localisation of  $\mathcal{U}$  with respect to the weak equivalences in  $\mathcal{U}$  and is denoted by  $\text{Ho } \mathcal{U}$ . The localisation functor of  $\text{Ho } \mathcal{U}$  is denoted by  $\Gamma = \Gamma^{\text{Ho } \mathcal{U}}: \mathcal{U} \rightarrow \text{Ho } \mathcal{U}$ .

Given an object  $X \in \text{Ob } \mathcal{M}$ , the category  $(X \downarrow \mathcal{M})$  of objects under  $X$  obtains a model category structure where a morphism in  $(X \downarrow \mathcal{M})$  is a weak equivalence resp. a cofibration resp. a fibration if and only if it is one in  $\mathcal{M}$ .

## Homotopies

A *cylinder* for an object  $X \in \text{Ob } \mathcal{M}$  consists of an object  $Z \in \text{Ob } \mathcal{M}$ , a cofibration  $\begin{pmatrix} \text{ins}_0 \\ \text{ins}_1 \end{pmatrix} = \text{ins} = \text{ins}^Z: X \amalg X \rightarrow Z$  and a weak equivalence  $s = s^Z: Z \rightarrow X$  such that  $\text{ins} s = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ .

Given parallel morphisms  $f, g: X \rightarrow Y$  in  $\mathcal{M}$ , we say that  $f$  is *cylinder homotopic* to  $g$ , written  $f \stackrel{\sim}{\sim} g$ , if there exists a cylinder  $Z$  for  $X$  and a morphism  $H: Z \rightarrow Y$  with  $\text{ins}_0 H = f$  and  $\text{ins}_1 H = g$ . In this case,  $H$  is said to be a *cylinder homotopy* from  $f$  to  $g$ . (In the literature, cylinder homotopy is also called left homotopy, cf. [5, ch. I, §1, def. 3, def. 4, lem. 1].) The relation  $\stackrel{\sim}{\sim}$  is reflexive and symmetric, but in general not transitive. Moreover,  $\stackrel{\sim}{\sim}$  is compatible with composition in  $\mathbf{Fib}(\mathcal{M})$ . We denote by  $\mathbf{Fib}(\mathcal{M})/\stackrel{\sim}{\sim}$  the quotient category of  $\mathbf{Fib}(\mathcal{M})$  with respect to the congruence generated by  $\stackrel{\sim}{\sim}$ .

## Quillen's homotopy category theorem

There are dual notions to fibrant objects, cylinders, cylinder homotopic  $\stackrel{\sim}{\sim}$ , the full subcategory of fibrant objects  $\mathbf{Fib}(\mathcal{M})$ , its quotient category  $\mathbf{Fib}(\mathcal{M})/\stackrel{\sim}{\sim}$  and its homotopy category  $\text{Ho } \mathbf{Fib}(\mathcal{M})$ , namely *cofibrant objects*, *path objects*, *path homotopic*  $\stackrel{\mathcal{P}}{\sim}$ , the full subcategory of cofibrant objects  $\mathbf{Cof}(\mathcal{M})$ , its quotient category  $\mathbf{Cof}(\mathcal{M})/\stackrel{\mathcal{P}}{\sim}$  and its homotopy category  $\text{Ho } \mathbf{Cof}(\mathcal{M})$ , respectively. Moreover, an object  $X \in \text{Ob } \mathcal{M}$  is said to be bifibrant if it is cofibrant and fibrant. On the full subcategory of bifibrant objects  $\mathbf{Bif}(\mathcal{M})$ , the relations  $\stackrel{\sim}{\sim}$  and  $\stackrel{\mathcal{P}}{\sim}$  coincide and yield a congruence. One writes  $\sim := \stackrel{\sim}{\sim} = \stackrel{\mathcal{P}}{\sim}$  in this case, and the quotient category is denoted by  $\mathbf{Bif}(\mathcal{M})/\sim$ . Moreover,  $\text{Ho } \mathbf{Bif}(\mathcal{M})$  is a localisation of  $\mathbf{Bif}(\mathcal{M})$  with respect to the weak equivalences in  $\mathbf{Bif}(\mathcal{M})$ .

Quillen's homotopy category theorem [5, ch. I, §1, th. 1] (cf. also [4, cor. 1.2.9, th. 1.2.10]) states that the various inclusion and localisation functors induce the following commutative diagram, where the functors labeled by  $\simeq$  are equivalences and the functor labeled by  $\cong$  is an isofunctor.

$$\begin{array}{ccccc}
\mathbf{Cof}(\mathcal{M})/\stackrel{\mathcal{P}}{\sim} & \longrightarrow & \text{Ho } \mathbf{Cof}(\mathcal{M}) & & \\
\uparrow & & \uparrow \simeq & \searrow \simeq & \\
\mathbf{Bif}(\mathcal{M})/\sim & \xrightarrow{\cong} & \text{Ho } \mathbf{Bif}(\mathcal{M}) & & \text{Ho } \mathcal{M} \\
\downarrow & & \downarrow \simeq & \nearrow \simeq & \\
\mathbf{Fib}(\mathcal{M})/\stackrel{\sim}{\sim} & \longrightarrow & \text{Ho } \mathbf{Fib}(\mathcal{M}) & & 
\end{array}$$

In this note, we treat the question whether the functors  $\mathbf{Fib}(\mathcal{M})/\stackrel{\sim}{\sim} \rightarrow \text{Ho } \mathbf{Fib}(\mathcal{M})$  and  $\mathbf{Cof}(\mathcal{M})/\stackrel{\mathcal{P}}{\sim} \rightarrow \text{Ho } \mathbf{Cof}(\mathcal{M})$  are faithful. By duality, it suffices to consider the first functor.

## The model category $\mathbf{mod}(\mathbb{Z}/4)$

The category  $\mathbf{mod}(\mathbb{Z}/4)$  of finitely generated modules over  $\mathbb{Z}/4$  is a Frobenius category (with respect to all short exact sequences), that is, there are enough projective and injective objects in  $\mathbf{mod}(\mathbb{Z}/4)$  and, moreover, these objects coincide (we call such objects bijective). Therefore  $\mathbf{mod}(\mathbb{Z}/4)$  carries a canonical model category structure (cf. also [4, sec. 2.2]): The cofibrations are the monomorphisms and the fibrations are the epimorphisms in  $\mathbf{mod}(\mathbb{Z}/4)$ . Every object in  $\mathbf{mod}(\mathbb{Z}/4)$  is bifibrant, and the weak equivalences are precisely the homotopy equivalences, where parallel morphisms  $f$  and  $g$  are homotopic if  $g - f$  factors over a bijective object in  $\mathbf{mod}(\mathbb{Z}/4)$ . That is, the weak equivalences in  $\mathbf{mod}(\mathbb{Z}/4)$  are the stable isomorphisms and the homotopy category of  $\mathbf{mod}(\mathbb{Z}/4)$  is isomorphic to the stable category of  $\mathbf{mod}(\mathbb{Z}/4)$ , cf. [2, ch. I, sec. 2.2].

We remark that every object in  $\mathbf{mod}(\mathbb{Z}/4)$  is isomorphic to  $(\mathbb{Z}/4)^{\oplus k} \oplus (\mathbb{Z}/2)^{\oplus l}$  for some  $k, l \in \mathbb{N}_0$ , and every bijective object is isomorphic to  $(\mathbb{Z}/4)^{\oplus k}$  for some  $k \in \mathbb{N}_0$ .

### §3 Faithfulness of the functor $\mathbf{Fib}(\mathcal{M})/\overset{\circ}{\sim} \rightarrow \mathbf{Ho} \mathbf{Fib}(\mathcal{M})$

We give a sufficient criterion for the functor under consideration to be faithful.

**Proposition.** If the model category  $\mathcal{M}$  is left proper and if  $w \amalg w$  is a weak equivalence for every weak equivalence  $w$  in  $\mathcal{M}$ , then  $\overset{\circ}{\sim}$  is a congruence on  $\mathbf{Fib}(\mathcal{M})$  and the canonical functor  $\mathbf{Fib}(\mathcal{M})/\overset{\circ}{\sim} \rightarrow \mathbf{Ho} \mathbf{Fib}(\mathcal{M})$  is faithful.

*Proof.* We suppose given fibrant objects  $X$  and  $Y$  and morphisms  $f, g: X \rightarrow Y$  with  $\Gamma f = \Gamma g$  in  $\mathbf{Ho} \mathbf{Fib}(\mathcal{M})$ . By [1, th. 1(ii)], there exists a weak equivalence  $w: X' \rightarrow X$  such that  $wf \overset{p}{\sim} wg$ . It follows that  $wf \overset{\circ}{\sim} wg$  by [5, ch. I, §1, dual of lem. 5], that is, there exists a cylinder  $Z'$  for  $X'$  and a cylinder homotopy  $H': Z' \rightarrow Y$  from  $wf$  to  $wg$ . We let

$$\begin{array}{ccc} X' \amalg X' & \xrightarrow[\approx]{w \amalg w} & X \amalg X \\ \text{ins}^{Z'} \downarrow & & \downarrow i \\ Z' & \xrightarrow[\approx]{w'} & Z \end{array}$$

be a pushout of  $w \amalg w$  along  $\text{ins}^{Z'}$ . By assumption,  $w \amalg w$  and  $w'$  are weak equivalences. Since  $(w \amalg w) \left( \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right) = \text{ins}^{Z'} s^{Z'} w$ , there exists a unique morphism  $s: Z \rightarrow X$  with  $\left( \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right) = is$  and  $s^{Z'} w = w' s$ . Then  $s$  is a weak equivalence since  $s^{Z'}$ ,  $w$  and  $w'$  are weak equivalences and therefore  $Z$  becomes a cylinder for  $X$  with  $\text{ins}^Z := i$  and  $s^Z := s$ . Moreover,  $(w \amalg w) \left( \begin{smallmatrix} f \\ g \end{smallmatrix} \right) = \text{ins}^{Z'} H'$  implies that there exists a unique morphism  $H: Z \rightarrow Y$  with  $\left( \begin{smallmatrix} f \\ g \end{smallmatrix} \right) = \text{ins}^Z H$  and  $H' = w' H$ . So in particular  $f \overset{\circ}{\sim} g$ .

$$\begin{array}{ccccc} X' \amalg X' & \xrightarrow[\approx]{w \amalg w} & X \amalg X & \xrightarrow{\left( \begin{smallmatrix} f \\ g \end{smallmatrix} \right)} & Y \\ \text{ins}^{Z'} \downarrow & & \downarrow \text{ins}^Z & & \parallel \\ Z' & \xrightarrow[\approx]{w'} & Z & \xrightarrow{H} & Y \\ s^{Z'} \downarrow \parallel & & \downarrow s^Z & & \\ X' & \xrightarrow[\approx]{w} & X & & \end{array}$$

Altogether, we have shown that morphisms in  $\mathbf{Fib}(\mathcal{M})$  represent the same morphism in  $\mathbf{Ho} \mathbf{Fib}(\mathcal{M})$  if and only if they are cylinder homotopic. In particular,  $\overset{\circ}{\sim}$  is a congruence on  $\mathbf{Fib}(\mathcal{M})$ .  $\square$

The following counterexample shows that the canonical functor  $\mathbf{Fib}(\mathcal{M})/\overset{\circ}{\sim} \rightarrow \mathbf{Ho} \mathbf{Fib}(\mathcal{M})$  is not faithful in general.

**Example.** We consider the category  $(\mathbb{Z}/4 \downarrow \mathbf{mod}(\mathbb{Z}/4))$  of finitely generated  $\mathbb{Z}/4$ -modules under  $\mathbb{Z}/4$  with the model category structure inherited from  $\mathbf{mod}(\mathbb{Z}/4)$ , cf. §2. All objects of  $(\mathbb{Z}/4 \downarrow \mathbf{mod}(\mathbb{Z}/4))$  are fibrant since all objects in  $\mathbf{mod}(\mathbb{Z}/4)$  are fibrant.

We study morphisms  $(\mathbb{Z}/4, 2) \rightarrow (\mathbb{Z}/4 \oplus \mathbb{Z}/2, (2 \ 0))$  in  $(\mathbb{Z}/4 \downarrow \mathbf{mod}(\mathbb{Z}/4))$ . We let  $(Z, t)$  be a cylinder of  $(\mathbb{Z}/4, 2)$  and we let  $H: (Z, t) \rightarrow (\mathbb{Z}/4 \oplus \mathbb{Z}/2, (2 \ 0))$  be a cylinder homotopy (from  $\text{ins}_0 H$  to  $\text{ins}_1 H$ ). Then we have a weak equivalence  $(Z, t) \rightarrow (\mathbb{Z}/4, 2)$  in  $(\mathbb{Z}/4 \downarrow \mathbf{mod}(\mathbb{Z}/4))$  and hence a weak equivalence  $Z \rightarrow \mathbb{Z}/4$  in  $\mathbf{mod}(\mathbb{Z}/4)$ . Thus  $Z$  is bijective and therefore we may assume that  $Z = (\mathbb{Z}/4)^{\oplus k}$ . Since  $\text{ins}_0$  and  $\text{ins}_1$  are morphisms from  $(\mathbb{Z}/4, 2)$  to  $(Z, t)$ , we have  $2\text{ins}_0 = t = 2\text{ins}_1$  and hence  $\text{ins}_0 \equiv_2 \text{ins}_1$  as morphisms from  $\mathbb{Z}/4$  to  $Z$ . But this implies that the second components of  $\text{ins}_0 H$  and  $\text{ins}_1 H$  are the same. In other words, we have shown that cylinder homotopic morphisms from  $(\mathbb{Z}/4, 2)$  to  $(\mathbb{Z}/4 \oplus \mathbb{Z}/2, (2 \ 0))$  coincide in the second component. It follows that the morphisms  $\left( \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right): (\mathbb{Z}/4, 2) \rightarrow (\mathbb{Z}/4 \oplus \mathbb{Z}/2, (2 \ 0))$  and  $\left( \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right): (\mathbb{Z}/4, 2) \rightarrow (\mathbb{Z}/4 \oplus \mathbb{Z}/2, (2 \ 0))$  in  $(\mathbb{Z}/4 \downarrow \mathbf{mod}(\mathbb{Z}/4))$  represent different morphisms in the quotient category  $\mathbf{Fib}((\mathbb{Z}/4 \downarrow \mathbf{mod}(\mathbb{Z}/4)))/\overset{\circ}{\sim}$ .

On the other hand, since  $\mathbb{Z}/4$  is bijective, the morphism  $2: \mathbb{Z}/4 \rightarrow \mathbb{Z}/4$  is a weak equivalence in  $\mathbf{mod}(\mathbb{Z}/4)$ , and therefore  $2: (\mathbb{Z}/4, 1) \rightarrow (\mathbb{Z}/4, 2)$  is a weak equivalence in  $(\mathbb{Z}/4 \downarrow \mathbf{mod}(\mathbb{Z}/4))$ . But  $2 \begin{pmatrix} 1 & 0 \end{pmatrix} = 2 \begin{pmatrix} 1 & 1 \end{pmatrix}$  as morphisms from  $(\mathbb{Z}/4, 1)$  to  $(\mathbb{Z}/4 \oplus \mathbb{Z}/2, \begin{pmatrix} 2 & 0 \end{pmatrix})$  in  $(\mathbb{Z}/4 \downarrow \mathbf{mod}(\mathbb{Z}/4))$ , so in particular  $\Gamma(2 \begin{pmatrix} 1 & 0 \end{pmatrix}) = \Gamma(2 \begin{pmatrix} 1 & 1 \end{pmatrix})$  and hence  $\Gamma \begin{pmatrix} 1 & 0 \end{pmatrix} = \Gamma \begin{pmatrix} 1 & 1 \end{pmatrix}$ .

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